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Fine Structures of Hyperbolic Diffeomorphisms

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In celebration of the 60th birthday of
David A. Rand

For

Maria Guiomar dos Santos Adrego Pinto

Bärbel Finkenstädt and the Rand kids: Ben, Tamsin, Rupert and Charlotte

Fernanda Amélia Ferreira and Flávio André Ferreira

Family and friends

Dedicated to Dennis Sullivan and Christopher Zeeman.

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Alberto Pinto
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Flávio Ferreira

Preface

The study of hyperbolic systems is a core theme of modern dynamics. On surfaces the theory of the fine scale structure of hyperbolic invariant sets and their measures can be described in a very complete and elegant way, and is the subject of this book, largely self-contained, rigorously and clearly written. It covers the most important aspects of the subject and is based on several scientific works of the leading research workers in this field.

This book fills a gap in the literature of dynamics. We highly recommend it for any Ph.D student interested in this area. The authors are well-known experts in smooth dynamical systems and ergodic theory.

Now we give a more detailed description of the contents:

Chapter 1. The Introduction is a description of the main concepts in hyperbolic dynamics that are used throughout the book. These are due to Bowen, Hirsch, Mañé, Palis, Pugh, Ruelle, Shub, Sinai, Smale and others. Stable and unstable manifolds are shown to be C^r foliated. This result is very useful in a number of contexts. The existence of smooth orthogonal charts is also proved. This chapter includes proofs of extensions to hyperbolic diffeomorphisms of some results of Mañé for Anosov maps.

Chapter 2. All the smooth conjugacy classes of a given topological model are classified using Pinto's and Rand's HR structures. The affine structures of Ghys and Sullivan on stable and unstable leaves of Anosov diffeomorphisms are generalized.

Chapter 3. A pair of stable and unstable solenoid functions is associated to each HR structure. These pairs form a moduli space with good topological properties which are easily described. The scaling and solenoid functions introduced by Cui, Feigenbaum, Fisher, Gardiner, Jiang, Pinto, Quas, Rand and Sullivan, give a deeper understanding of the smooth structures of one and two dimensional dynamical systems.

Chapter 4. The concept of self-renormalizable structures is introduced. With this concept one can prove an equivalence between two-dimensional hyperbolic sets and pairs of one-dimensional dynamical systems that are renormalizable (see also Chapter 12). Two C^{1+} hyperbolic diffeomorphisms that are smoothly conjugate at a point are shown to be smoothly conjugate. This extends some results of de Faria and Sullivan from one-dimensional dynamics to two-dimensional dynamics.

Chapter 5. A rigidity result is proved: if the holonomies are smooth enough, then the hyperbolic diffeomorphism is smoothly conjugate to an affine model. This chapter extends to hyperbolic diffeomorphisms some of the results of Avez, Flaminio, Ghys, Hurder and Katok for Anosov diffeomorphisms.

Chapter 6. An elementary proof is given for the existence and uniqueness of Gibbs states for Hölder weight systems following pioneering works of Bowen, Paterson, Ruelle, Sinai and Sullivan.

Chapter 7. The measure scaling functions that correspond to the Gibbs measure potentials are introduced.

Chapter refsmeasures. Measure solenoid and measure ratio functions are introduced. They determine which Gibbs measures are realizable by C^{1+} hyperbolic diffeomorphisms and by C^{1+} self-renormalizable structures.

Chapter 9. The cocycle-gap pairs that allow the construction of all C^{1+} hyperbolic diffeomorphisms realizing a Gibbs measure are introduced.

Chapter 10. A geometric measure for hyperbolic dynamical systems is defined. The explicit construction of all hyperbolic diffeomorphisms with such a geometric measure is described, using the cocycle-gap pairs. The results of this chapter are related to Cawley's cohomology classes on the torus.

Chapter 11. An eigenvalue formula for hyperbolic sets on surfaces with an invariant measure absolutely continuous with respect to the Hausdorff measure is proved. This extends to hyperbolic diffeomorphisms the Livšic-Sinai eigenvalue formula for Anosov diffeomorphisms preserving a measure absolutely continuous with respect to Lebesgue measure. Also given here is an extension to hyperbolic diffeomorphisms of the results of De la Llave, Marco and Moriyon on the eigenvalues for Anosov diffeomorphisms.

Chapter 12. A one-to-one correspondence is established between C^{1+} arc exchange systems that are C^{1+} fixed points of renormalization and C^{1+} hyperbolic diffeomorphisms that admit an invariant measure absolutely continuous with respect to the Hausdorff measure. This chapter is related to the work of Ghys, Penner, Rozzy, Sullivan and Thurston. Further, there are connections with the theorems of Arnold, Herman and Yoccoz on the rigidity of circle diffeomorphisms and Denjoy's Theorem. These connections are similar to the ones between Harrison's conjecture and the investigations of Kra, Norton and Schmeling.

Chapter 13. Pinto's golden tilings of the real line are constructed (see Pinto's and Sullivan's d-adic tilings of the real line in the Appendix C). These golden tilings are in one-to-one correspondence with smooth conjugacy classes of golden diffeomorphisms of the circle that are fixed points of renormalization, and also with smooth conjugacy classes of Anosov diffeomorphisms with an invariant measure absolutely continuous with respect to the Lebesgue measure. The observation of Ghys and Sullivan that Anosov diffeomorphisms on the torus determine circle diffeomorphisms having an associated renormalization operator is used.

Chapter 14. Thurston's pseudo-Anosov affine maps appear as periodic points of the geodesic Teichmüller flow. The works of Masur, Penner, Thurston and Veech show a strong link between affine interval exchange maps and pseudo-Anosov affine maps. Pinto's and Rand's pseudo-smooth structures near the singularities are constructed so that the pseudo-Anosov maps are smooth and have the property that the stable and unstable foliations are uniformly contracted and expanded by the pseudo-Anosov dynamics. Classical results for hyperbolic dynamics such as Bochi-Mañé and Viana's duality extend to these pseudo-smooth structures. Blow-ups of these pseudo-Anosov diffeomorphisms are related to Pujals' non-uniformly hyperbolic diffeomorphisms.

Appendices. Various concepts and results of Pinto, Rand and Sullivan for one-dimensional dynamics are extended to two-dimensions. Ratio and cross-ratio distortions for diffeomorphisms of the real line are discussed, in the spirit of de Melo and van Strien's book.

Rio de Janeiro, Brazil
July 2008

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